Automata Theory and Formal Grammars: Lecture 2

Deterministic and Nondeterministic Finite Automata

Deterministic and Nondeterministic Finite Automata

Last Time

- Sets Theory (Review?)
- Logic, Proofs (Review?)
- Words, and operations on them: $w_1 \circ w_2, w^i, w^*, w^+$
- Languages, and operations on them: $L_1 \circ L_2, L^i, L^*, L^+$

Today

- Deterministic Finite Automata (DFAs) and their languages
- Closure properties of DFA languages (the product construction)
- Nondeterministic Finite Automata (NFAs) and their languages
- Relating DFAs and NFAs (the subset construction)

Fibonacci as a Recursively Defined Set

The n^{th} Fibonacci number $f(n)$:

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 1 \\ f(n) &= f(n-1) + f(n-2), \text{ for } n \geq 2 \end{aligned}$$

As a recursively defined set (relation)

$$\begin{aligned} F_0 &= \emptyset \\ F_{i+1} &= \{\langle 0, 0\rangle, \langle 1, 1\rangle\} \\ &\cup \left\{ \langle n, f_{n_1} + f_{n_2}\rangle \;\middle|\; \begin{array}{ll} \langle n_1, f_{n_1}\rangle \in F_i & \text{and} \\ \langle n_2, f_{n_2}\rangle \in F_i & \text{and} \\ n \;=\; n_1 + 1 \;=\; n_2 + 2 & \end{array} \right\} \end{aligned}$$

Fibonacci as a Recursively Defined Set

$$
\begin{aligned}
F_0 &= \emptyset \\
F_{i+1} &= \{\langle 0,0\rangle, \langle 1,1\rangle\} \\
&\cup \left\{ \langle n, f_{n_1} + f_{n_2}\rangle \;\middle|\; \begin{array}{ll} \langle n_1, f_{n_1}\rangle \in F_i & \text{and} \\ \langle n_2, f_{n_2}\rangle \in F_i & \text{and} \\ n = n_1 + 1 = n_2 + 2 & \end{array} \right\}
\end{aligned}
$$

For example:

$$
\begin{aligned}
F_0 &= \emptyset \\
F_1 &= \{\langle 0,0\rangle, \langle 1,1\rangle\} \\
F_2 &= \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,1\rangle\} \\
F_3 &= \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,1\rangle, \langle 3,2\rangle\} \\
F_4 &= \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,1\rangle, \langle 3,2\rangle, \langle 4,3\rangle\} \\
F_5 &=
\end{aligned}
$$

Conventions

- Σ is an arbitrary alphabet. (In examples, Σ should be clear from context.)
- The variables a–e range over letters in Σ.
- The variables u–z range over words over Σ^*.
- The variables p–q range over states in Q.

Recall

For any string w and language L:

$$w \circ \varepsilon = w \qquad = \varepsilon \circ w \tag{1}$$

$$L \circ \{\varepsilon\} = L \qquad = \{\varepsilon\} \circ L \tag{2}$$

$$L^* = \{\varepsilon\} \cup L \circ L^* \tag{3}$$

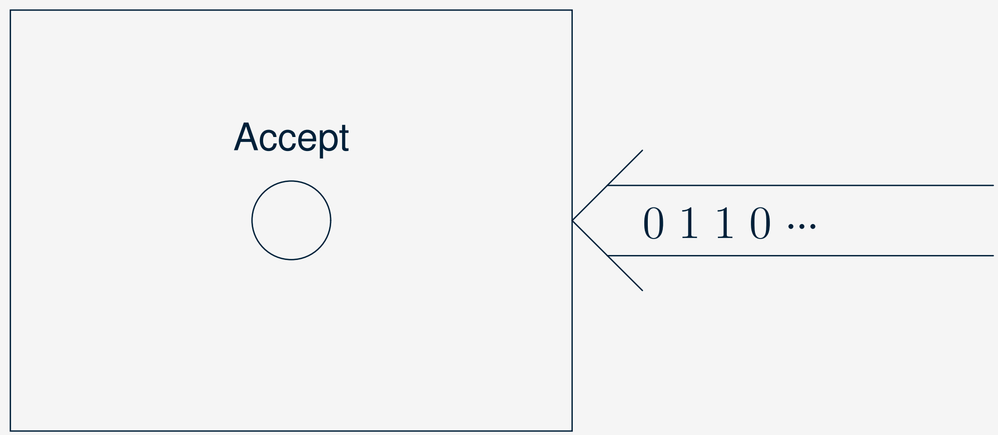
L^* is closed with respect to concatenation, for any L:

if $u \in L^*$ and $v \in L^*$ then $u \circ v \in L^*$

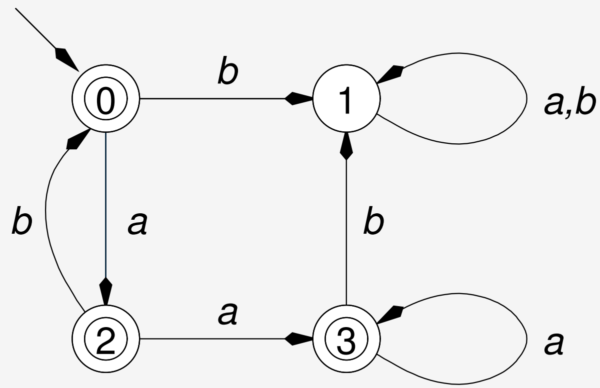
Finite Automata

... are "machines" for recognizing languages!

- They process input words a symbol at a time.
- An "accept light" flashes if the symbols read in so far are "OK".



Formal Definition of Finite Automata



Definition A finite automaton (DFA) is a quintuple $\langle Q, \Sigma, q_0, \delta, A \rangle$ where:

- Q is a finite non-empty set of states;
- Σ is an alphabet;
- $q_0 \in Q$ is the start state;
- $\delta : Q \times \Sigma \rightarrow Q$ is the transition function; and
- $A \subseteq Q$ is the set of accepting (final) states.

DFA Acceptance

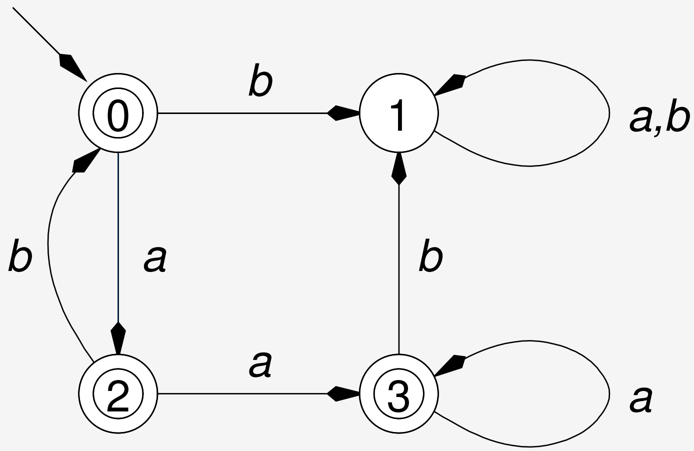
Given a DFA $M = \langle Q, \Sigma, q_0, \delta, A \rangle$ and word $w \in \Sigma^*$:

- M should accept w if in processing w a symbol at a time, M goes to an accepting state.
- To formalize this we define a function

  $$\delta^* : Q \times \Sigma^* \to Q$$

  $\delta^*(q, w)$ should be the state reached from q after processing w.
- How to define δ^*?

Example of δ^*



$$
\begin{aligned}
\delta^*(0, aab) &= \delta^*(\delta(0,a), ab) = \delta^*(2, ab) \\
&= \delta^*(\delta(2,a), b) = \delta^*(3, b) \\
&= \delta^*(\delta(3,b), \varepsilon) = \delta^*(1, \varepsilon) \\
&= 1
\end{aligned}
$$

What is $\delta^*(0, abaa)$?

Definition of δ^*

Definition Let $M = \langle Q, \Sigma, q_0, \delta, A \rangle$ be a DFA. Then $\delta^* : Q \times \Sigma^* \to Q$ is defined recursively:

$$\delta^*(q, w) = \begin{cases} q & \text{if } w = \varepsilon \\ \delta^*(\delta(q, a), w') & \text{if } w = aw' \text{ and } a \in \Sigma \end{cases}$$

$\delta^*(q, w) = q'$ if q' the state reached by processing w, starting from q.

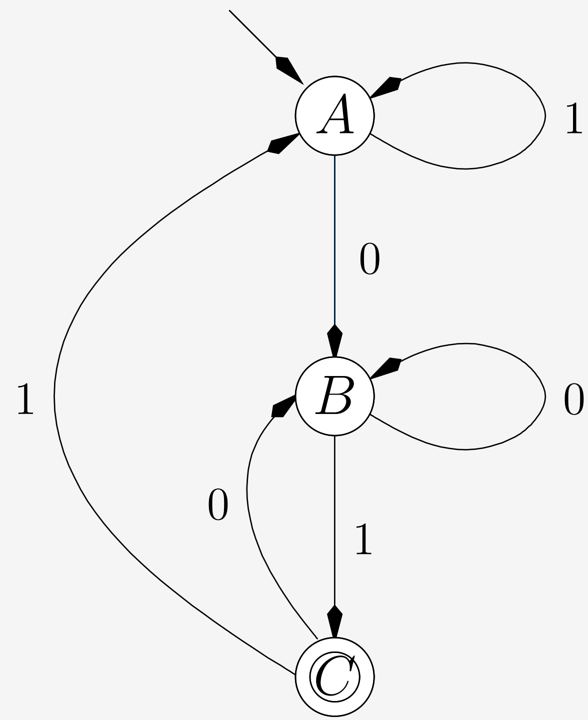
Language of a Finite Automaton

A DFA accepts a word if it reaches an accepting state after "consuming" the word.

Definition Let $M = \langle Q, \Sigma, q_0, \delta, A \rangle$ be a DFA.

- M accepts $w \in \Sigma^*$ if $\delta^*(q_0, w) \in A$.
- $\mathcal{L}(M) = \{\, w \in \Sigma^* \mid M \text{ accepts } w \,\}$ is the language accepted by M.

Example: DFA for $\{ w \in \{0,1\}^* \mid w \text{ ends in } 01 \}$



Example: DFA for Valid Binary Numbers

- Must contain at least one digit.
- No leading 0s.