L-domains and Lossless Powerdomains

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Abstract

The category of L-domains was discovered by A. Jung while solving the problem of finding maximal cartesian closed categories of algebraic CPO's and continuous functions. In this note we analyse properties of the lossless powerdomain construction, that is closed on the algebraic L-domains. The powerdomain is shown to be isomorphic to a collection of subsets of the domain on which the construction was done. The proof motivates a certain finiteness condition on the inconsistency relations of elements. It is shown that all algebraic CPO's D whose basis $B(D)$ has property M satisfy the condition. In particular, the coherent L- domains satisfy the condition.

1 Introduction

Recent work by A. Jung and C. Gunter shows that the L-domains discovered independently by A. Jung[6] and T. Coquand[1] form an interesting cartesian closed category. P. Buneman proposed the lossless powerdomain construction that is closed on L-domains. Buneman's construction was based on intuitions from databases. We were studying the construction as a possible candidate for modelling oracleisable indeterminacy. The present paper investigates some purely mathematical questions that arose from the study.

A representation theorem for the lossless powerdomain is proved for the lossless powerdomain in Section 3. The proof suggests some additional natural conditions on L-domains. The proof uses the notion of finite separability, the ability to separate elements of the sets that constitute the powerdomain by disjoint basic Scott open sets (this is made precise in Section 3). This suggests that a natural condition to consider is that the inconsistency relation have finite witnesses. Section 4 discusses this notion of finite inconsistency. Discussions with C. Gunter and E. Gunter revealed the relationship of this property to the coherent L-domains[4].

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2 Preliminaries

This section outlines the basic definitions and facts that are used in the note.

A subset X of a partially ordered set is directed iff it is non-empty and every pair of elements in X has an upper bound in X. A partial order D is said to be (directed) complete if every directed subset X of D has a least upper bound in D. We shall only consider CPO's with a least element, which will be denoted $\perp$. An element d of D is said to be compact if for every directed set D such that $d \sqsubseteq \bigsqcup D$, there is an element $x \in D$ such that $d \sqsubseteq x$. The set of all elements of D greater than a compact element d is denoted by $d\uparrow$. The set of all elements of D less than an element x is denoted by $x\downarrow$. A CPO D is said to be algebraic if every element is the lub of a directed set of compact elements. The set of compact elements of an algebraic CPO D is denoted by $B(D)$. Step functions of the form $d \searrow e$ where d and e are compact elements of D and E respectively are compact elements of the function space $D \rightarrow E$. The set of minimal upper bounds of a subset A of D that are below elements of a subset B of D is denoted by $mub_A\,(B)$.

An algebraic CPO D is said to be an algebraic L-domain when any of the following equivalent conditions hold [6].

1. For each $x \in D$, the set $x \downarrow \bigcap B(D)$ is a $\vee$-semilattice with smallest element.
2. For each upper bound x of a finite subset A of $B(D)$, there is a unique minimal upper bound of A below x.
3. For any finite subset A of $B(D)$, $mub_A\,(D) = mub_{(mub_A\,(D))}\,(D)$

3 The Lossless Powerdomain

Peter Buneman discovered the lossless powerdomain construction [3] that is closed on L-domains.

Definition 1 *Let D be an L-domain. Define the preorder $P_L(D)$as follows:*

$$|P_L(D)| = \{(e_1 \ldots e_n)\; |(\forall 1 \leq i \leq n)\; [e_i \in B(D)] \;\wedge\; (\forall 1 \leq i, j \leq n)\; [i \neq j \Longrightarrow mub_{(e_i,\, e_j)}\,(D) = \phi]\}$$

and the elements of $|P_L(D)|$ are ordered by the Egli-Milner ordering, $\sqsubseteq_{EM}$.

Actually $P_L(D)$ is a partial order. The lossless powerdomain $\overline{P_L(D)}$ is constructed by ideal-completion of $P_L(D)$.

Lemma 1 $\overline{P_L(D)}$*is an L-domain if D is.*

Proof: It suffices to prove that $(\forall c \in P_L(D))\; [c \downarrow \bigcap P_L(D)]$ is an $\vee$- semilattice under the partial order $\sqsubseteq_{EM}$. Let

$$a = (e_1 \ldots e_n),\; b = (d_1 \ldots d_m),\; c = (f_1 \ldots f_p),\; a, b \sqsubseteq_{EM} c$$

Then, $\bigsqcup\{a,b\}$ under c is given by,

$$\bigsqcup\{a,b\} \;=\; \bigcup\{mub_{(e_i,\,d_j)}\,(c) | 1 \le i \le n, 1 \le j \le m,\; e_i \uparrow \bigcap d_j \uparrow \bigcap c \neq \phi\}$$

The verification that the above definition is correct is quite easy. ∎

It is shown that the ideals in the lossless powerdomain are representatives of their fringe sets. Fringe sets are the sets generated by following the partial order arrows among the elements of the ideal. The following definition captures the idea of "following arrows".

Definition 2 *A* GENERATOR *d over a ideal $I \in \overline{P_L(D)}$ is a function $d:\; I \rightarrow D$ such that :*

- *$i \in I \Longrightarrow d\,(i) \in i$*
- *$(\, i \in I \wedge\; j \in I\; \wedge i \sqsubseteq_{EM} j) \Longrightarrow d\,(i) \sqsubseteq d\,(j)$*

The following lemma is an easy consequence of the definition.

Lemma 2 *If d is a* GENERATOR *over I, $d\,(I)$ is a directed set in D.*

Definition 3 *Let $I \in \overline{P_L(D)}$. The set generated by I is*

$$S_I = \{\, \bigsqcup d\,(I) |\; d \text{ is a Generator on } I\}$$

Since the lossless powerdomain embodies finite branching only, one expects the sets generated to be Scott-compact. The proof requires the following lemma, that is a consequence of Rudin's lemma [2].

Lemma 3 *Let $I \in \overline{P_L(D)}$. Let $\tilde{I}$ be cofinal in I. Let P be a predicate defined on D such that*

1. *$(\forall i \in\; \tilde{I})\; (\exists e \in\; i)\; [P(e)]$*
2. *$(d \in i \in \tilde{I}\; \wedge e \in j \in \tilde{I}\; \wedge i \sqsubseteq_{EM} j\; \wedge d \sqsubseteq e\; \wedge P(e)) \Longrightarrow P(d)$*

Then there is a generator d over I such that $(\forall i \in \tilde{I})\; [P\;(d\;(i))]$.

Lemma 4 *If $I \in \overline{P_L(D)}$, S_I is non-empty and Scott-compact.*

Proof:

1. Define a predicate P on D by

   $$P(e) \Longleftrightarrow\; (\exists i \in I)\; [e \in i]$$

   From lemma 3, we get a generator on D. Hence S_I is non-empty.

2. Let $\{x_\alpha\}$ be a net in S_I. Define a predicate P on D as follows:

$$P(e) \Longleftrightarrow (\exists i \in I)\ [e \in i\ \wedge\ e \uparrow \bigcap \{x_\alpha\} \text{ is cofinal }]$$

From lemma 3 , we get a generator d. Consider $\bigsqcup d\ (I)$. Any neighbourhood $e \uparrow$ of $\bigsqcup d\ (I)$ has non-empty intersection with $d\ (I)$, and consequently is cofinal in $\{x_\alpha\}$. Hence $\bigsqcup d\ (I)$ is an accumulation point of $\{x_\alpha\}$ ∎

However, not all non-empty Scott-compact sets are generated by some ideal in the lossless powerdomain. Consider the L-domain in Fig 1. The set $\{x, y\}$ cannot be generated by any ideal in $\overline{P_L(D)}$even though the set $\{x, y\}$ is finite and hence compact in the Scott Topology. This observation motivates the following definition.

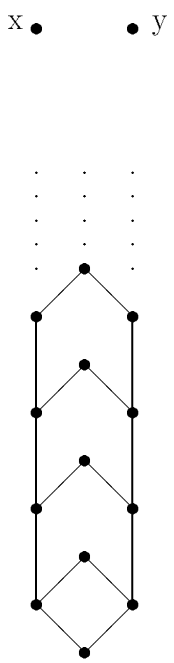

Fig. 1

Definition 4 *$S \subseteq D$ is* FINITELY SEPARABLE *if*

$$(\forall S_{fin} = \{x_1 \dots x_n\},\ S_{fin} \subseteq S)\ (\exists\ (e_1 \dots e_m) \in P_L(D))\ [n \leq m\ \wedge\ (\forall 1 \leq i \leq n)(e_i \sqsubseteq x_i)\ \wedge\ (e_1 \dots e_m) \sqsubseteq_{EM} S]$$

The following lemma is an easy consequence of the above definition.

Lemma 5 *If $I \in \overline{P_L(D)}$, S_I is* FINITELY SEPARABLE.

It turns out that one can generate all *FINITELY SEPARABLE*, Scott-compact sets.

Definition 5 *Let D be an L-domain. Then Ψ_D is the set of all Scott-compact finitely-separable subsets of D ordered by $\sqsubseteq_{EM}$*

Note that Ψ_D is a partial order. Now, we define maps between $\overline{P_L(D)}$and Ψ_D in a natural manner, and show that the definitions do indeed constitute an order isomorphism between the partial orders, giving us the required representation theorem.

Lemma 6 *Define* $\Phi : \overline{P_L(D)} \rightarrow \Psi_D$ *by*

$$\Phi\ (I)\ = S_I$$

Then, Φ *is monotone.*

Proof: Let $I_1 \in \overline{P_L(D)}, I_2 \in \overline{P_L(D)}, I_1 \subseteq I_2$. Let $\Phi\ (I_1) = S_{I_1}, \Phi\ (I_2) = S_{I_2}$.

- Let $x \in S_{I_1}$
  $\Longrightarrow (\exists d)\ [\bigsqcup d\ (I_1) = x]$, where d is generator on I_1. Define

  $$\tilde{I_2} = \{i |\ i \in I_2\ \wedge (\exists j \in I_1)\ [j \sqsubseteq_{EM} i]$$

  Note that $\tilde{I_2}$ is cofinal in I_2. Define predicate P by

  $$P(e) \Longleftrightarrow (\exists i,\ j)\ [i \in \tilde{I_2}\ \wedge\ e \in i\ \wedge\ j \in I_1\ \wedge\ d\ (j) \sqsubseteq e\ \wedge \tilde{e} \in i\ \wedge i \in \tilde{I_2}]$$

  The generator $\tilde{d}$ on I_2 given by lemma 4 satisfies $x \sqsubseteq \bigsqcup \tilde{d}$
  $\Longrightarrow (\forall x \in S_{I_1})\ (\exists y \in S_{I_2})\ [x \sqsubseteq y]$

- Let $x \in S_{I_2}$
  $\Longrightarrow (\exists \tilde{d})\ [\bigsqcup \tilde{d}\ (I_2) = x]$, where $\tilde{d}$ is a generator on I_2. Restriction of $\tilde{d}$ to I_1 gives a generator d on I_2 such that $d\ (I_1) \subseteq \tilde{d}\ (I_2)$. Hence, we have $\bigsqcup d\ (I_1) \sqsubseteq \bigsqcup \tilde{d}\ (I_2)$. Hence, we have

  $$(\forall y \in S_{I_2})\ (\exists x \in S_{I_1})\ [x \sqsubseteq y] \quad \blacksquare$$

Lemma 7 *Let* $S \in \Psi_D,\ (e_1 \ldots e_n) \sqsubseteq_{EM} S,\ (d_1 \ldots d_m) \sqsubseteq_{EM} S$. *Then*

$$(\forall 1 \leq i \leq n,\ 1 \leq j \leq m)\ [mub_{(e_i,\, d_j)}\ (S) \text{ is finite }]$$

Proof: Let $1 \leq i \leq n,\ 1 \leq j \leq m$. Consider the basic open cover of S consisting of

- $\tilde{e} \uparrow,\ \tilde{e} \in mub_{(e_i,\, d_j)}\ (S)$
- $d_{\tilde{j}} \uparrow,\ \tilde{j} \neq j,\ 1 \leq \tilde{j} \leq m$
- $e_{\tilde{i}} \uparrow,\ \tilde{i} \neq i,\ 1 \leq \tilde{i} \leq n$

This open cover has a finite subcover. Since all members of $(e_1, \ldots e_n)$ and $(d_1, \ldots d_m)$ are pairwise inconsistent, we deduce that $mub_{(e_i,\, d_j)}\ (S)$ is finite. $\blacksquare$

Lemma 8 *Define* $\tilde{\Phi} : \Psi_D \rightarrow \overline{P_L(D)}$ *by*

$$\tilde{\Phi}\ (S)\ = \{\ (e_1 \ldots e_m) |\ (e_1 \ldots e_m) \sqsubseteq_{EM} S\ \wedge (e_1 \ldots e_m) \in P_L(D)\}$$

Then, $\tilde{\Phi}$ *is monotone.*

Proof:

- We have to first show that $\tilde{\Phi}$ is well defined. Let $S \in \Psi_D$. Then we have
  - $\{\perp\} \in \tilde{\Phi}\ (S)$. So, $\tilde{\Phi}\ (S)$ is non-empty.
  - $(e_1 \ldots e_n) \sqsubseteq_{EM} S,\ (d_1 \ldots d_m) \sqsubseteq_{EM} S \Longrightarrow$
    $\bigcup\{\{mub_{(e_i,\ d_j)}\ (S)\}\ |1 \leq i \leq n,\ 1 \leq j \leq m,\ e_i \uparrow \bigcap d_j \uparrow \bigcap S \neq \phi\} \sqsubseteq_{EM} S$.
    Hence, $\tilde{\Phi}\ (S)$ is directed.
  - $\tilde{\Phi}\ (S)$ is obviously downward closed.

  Hence, $\tilde{\Phi}\ (S)$ is an element of $\overline{P_L(D)}$.
- $S_1 \sqsubseteq_{EM} S_2 \Longrightarrow \tilde{\Phi}\ (S_1) \subseteq \tilde{\Phi}\ (S_2)$. Hence, $\tilde{\Phi}$ is monotone. ∎

Lemma 9 $\tilde{\Phi} \circ \Phi\ = Id$

Proof: Note that it suffices to prove that

$$(\forall i \in P_L(D))\ [i \in I \Longleftrightarrow i \sqsubseteq_{EM} S_I]$$

The 'if' part is obvious. For the reverse direction, consider

$(e_1 \ldots e_m) \in P_L(D)\ \wedge\ (e_1 \ldots e_m) \sqsubseteq_{EM} S_I$
$\Longrightarrow (\exists x_1 \ldots x_m \in S_I)\ (\forall 1 \leq k \leq m)\ [e_k \sqsubseteq x_k]$

Let $d_1 \ldots d_m$ be the generators on I corresponding to $x_1 \ldots x_m$. Since $d_k\ (I)$ is a directed set for all $1 \leq k \leq m$, we have

$$(\forall 1 \leq k \leq m)\ (\exists \tilde{e_k}, i_k)\ [\tilde{e_k} \in i_k \in I\ \wedge d_k\ (i_k) = \tilde{e_k}\ \wedge e_k \sqsubseteq \tilde{e_k}\ ]$$

Since I is directed, $(\exists i \in I)\ [i_1 \ldots i_m \sqsubseteq_{EM} i]$. Note that $(\forall 1 \leq i \leq m)\ (\exists \tilde{e_i} \in i)\ [e_i \sqsubseteq \tilde{e_i}]$. Define

$$\tilde{I} = \{j |\ j \in I \wedge i \sqsubseteq_{EM} j\}$$

Note that $\tilde{I}$ is cofinal in I.

- We have

$$(\forall j \in \tilde{I})\ (\forall 1 \leq k \leq m)\ (\exists \tilde{e}_k \in j)\ [e_k \sqsubseteq \tilde{e}_k]$$

- We need to prove that $(\exists j \in \tilde{I})\ (\forall \tilde{e} \in j)\ (\exists 1 \leq k \leq m)\ [e_k \sqsubseteq \tilde{e}]$ Suppose not. Define a predicate P by:

$$P(e) \Longleftrightarrow (\exists j \in \tilde{I})\ [e \in j\ \wedge e_1 \not\sqsubseteq e \ldots e_m \not\sqsubseteq e]$$

P satisfies the conditions of lemma 3. The generator yielded by lemma 3 gives an element in S_I that is not greater than any of $e_1 \ldots e_m$. This is a contradiction since $(e_1 \ldots e_m) \sqsubseteq_{EM} S_I$

Hence, we have an element of I above $(e_1 \ldots e_m)$. The result follows by downward closure of I. ∎

Lemma 10 $\Phi \circ \tilde{\Phi} = Id$

Proof: Let $I = \tilde{\Phi}\,(S)$, where $S \in \Psi_D$.

1. Let d be a generator over I. We shall prove by contradiction that $(\exists x \in S)\,[\bigsqcup d\,(I) \sqsubseteq x]$. Suppose not. We have the cover

   $$\{e \uparrow \mid (\exists i \in I)\,[e \in i \;\wedge\; e \neq d\,(i)]\}$$

   Since S is Scott-compact, there is a finite sub-cover

   $$\{e_k \uparrow \mid (\exists i_k \in I)\,[e_k \in i_k \;\wedge\; e \neq d\,(i_k) \;\wedge\; 1 \leq k \leq n]\}$$

   Since I is directed,

   $(\exists i \in I)\,[i_1 \ldots i_k \sqsubseteq_{EM} i]$
   $\Longrightarrow \{e \uparrow \mid e \in (i - \{d\,(i)\})\}$ is a cover of S

   This is a contradiction, since $i \sqsubseteq_{EM} S$ means that there is an element in S greater than $d\,(i)$

2. Let $x \in S$. From definition of $\tilde{\phi}\,(S)$,

   $$(\forall i \in \tilde{\phi}\,(S))\,(\exists e_i \in i)\,[e_i \sqsubseteq x]$$

   Also, the e_i's are unique and form a directed set. Hence, a generator d_x can be defined in the obvious way such that $\bigsqcup d_x\,(I) \sqsubseteq x$.

3. Let $x, y \in S \;\wedge x \neq y$. Then from finite separability of S $d_x \neq d_y$, where d_x, d_y are the generators on I defined as above.

4. Now we shall show that $\bigsqcup d_x\,(I) = x$. Let $e \sqsubseteq x$. Consider the cover

   $$\{b \uparrow \mid b \in i \in I,\; b \neq d_x\,(i)\} \bigcup \{e \uparrow\}$$

   Note that the above is a cover because of finite separability. Since S is Scott-compact, we have a finite sub-cover

   $$\{b_k \uparrow \mid b_k \in i_k \in I,\; b_k \neq d_x\,(i_k),\; 1 \leq k \leq n\} \bigcup \{e \uparrow\}$$

Since I is directed, $(\exists i \in I)\ [i_1 \ldots i_n \sqsubseteq_{EM} i]$. Hence, we note that

$$\{b \uparrow \mid b \in i,\ b \neq d_x\ (i)\}\ \bigcup \{e \uparrow\}$$

is a cover. Hence, we have

$$(\forall y \in S)\ [d\ (i) \sqsubseteq y \Longrightarrow e \sqsubseteq y]$$

Hence, we deduce that we have the open cover consisting of

- $\tilde{e}$, where $\tilde{e} \in i - \{d\ (i)\}$
- $\tilde{m}$, where $\tilde{m} \in mub_{(d\ (i),\, e)}\ (S)$

The above open cover has a finite sub-cover. Hence we deduce that $mub_{(d\ (i),\, e)}\ (S)$ is finite. Hence $\tilde{i}$ defined as

$$\tilde{i} = mub_{(d\ (i),\, e)}\ (S) \bigcup (i - \{d\ (i)\}$$

satisifies $\tilde{i} \sqsubseteq_{EM} S\ \wedge \tilde{i} \in P_L(D)$. Hence, $e \sqsubseteq \bigsqcup d_x\ (I)$

The above shows that the generators on I generate precisely the elements of S. Hence,

$$\Phi \circ \tilde{\Phi} = Id. \quad \blacksquare$$

Theorem 1 *$\overline{P_L(D)}$ is isomorphic to Ψ_D*

4 Finitely Detectable Inconsistency

Consider the elements $x,\ y$ in Figure 1. Every pair of finite elements $e_x,\ e_y$ below $x,\ y$ respectively have upper-bounds. However $x,\ y$ do not have an upper bound. One might demand that the inconsistency relation have finite witnesses to make it continuous. The above discussion motivates the following definition.

Definition 6 *An algebraic CPO D is said to have property FI (for finitely detectable inconsistency) if*
$(\forall x,\ y \in D)\ [mub_{(x,\, y)}\ (D) = \phi \Longrightarrow (\exists e_x,\ e_y\ compact)$
$[e_x \sqsubseteq x\ \wedge e_y \sqsubseteq y\ \wedge\ mub_{(e_x,\, e_y)}\ (D) = \phi;]$

It is easy to check that all Scott domains have the above property. C. Gunter observed that all coherent L-domains [4] have property FI. The following lemma due to A. Jung [5] enables us to prove a stronger result.

Lemma 11 *An algebraic CPO D is Lawson-compact if and only if $B\ (D)$ has property M.*

Lemma 12 *Let D be an algebraic CPO such that every finite subset of $B(D)$ has a complete finite set of minimal upper bounds. (i.e $B(D)$ has Property M). Then, D has property FI.*

Proof: Let $x,\ y \in D$. Let $\{d_j |\ j \in I_x\}$ and $\{e_i |\ i \in I_y\}$ be the compact elements approximating x, y respectively, where I_x and I_y are index sets. Furthermore, let us assume that

$$(\forall d_j, e_i)\ [j \in I_x\ \wedge\ i \in I_y \Longrightarrow mub_{(d_j,\, e_i)}(D) \neq \phi]$$

Let $C = \{\{e_i \uparrow\} |\ i \in I_y\ \} \ \bigcup\ \{\{d_j \uparrow\} |\ j \in I_x\}$. Then C is a collection of closed sets in the Lawson topology on D satisfying the finite intersection condition. Result follows from the compactness of the Lawson Topology on D. ∎

In particular all SFP objects D have property FI.

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