3.2 Binary Search Trees

- BSTs
- ordered operations
- deletion
BSTs
- ordered operations
- deletion
Binary search trees

**Definition.** A BST is a *binary tree in symmetric order.*

A binary tree is either:
- Empty.
- Two disjoint binary trees (left and right).

**Symmetric order.** Each node has a key, and every node’s key is:
- Larger than all keys in its left subtree.
- Smaller than all keys in its right subtree.
BST representation in Java

Java definition. A BST is a reference to a root Node.

A Node is comprised of four fields:

- A Key and a Value.
- A reference to the left and right subtree.

```java
private class Node {
    private Key key;
    private Value val;
    private Node left, right;
    public Node(Key key, Value val) {
        this.key = key;
        this.val = val;
    }
}
```

Key and Value are generic types; Key is Comparable.
public class BST<Key extends Comparable<Key>, Value> {
    private Node root;

    private class Node {
        /* see previous slide */
    }

    public void put(Key key, Value val) {
        /* see next slides */
    }

    public Value get(Key key) {
        /* see next slides */
    }

    public void delete(Key key) {
        /* see next slides */
    }

    public Iterable<Key> iterator() {
        /* see next slides */
    }
}
BST search and insert demo
**BST search: Java implementation**

**Get.** Return value corresponding to given key, or `null` if no such key.

```java
public Value get(Key key)
{
    Node x = root;
    while (x != null)
    {
        int cmp = key.compareTo(x.key);
        if      (cmp  < 0) x = x.left;
        else if (cmp  > 0) x = x.right;
        else if (cmp == 0) return x.val;
    }
    return null;
}
```

**Cost.** Number of compares is equal to 1 + depth of node.
BST insert

Put. Associate value with key.
Search for key, then two cases:
• Key in tree ⇒ reset value.
• Key not in tree ⇒ add new node.
BST insert: Java implementation

Put. Associate value with key.

```java
public void put(Key key, Value val)
{  root = put(root, key, val);  }

private Node put(Node x, Key key, Value val)
{
    if (x == null) return new Node(key, val);
    int cmp = key.compareTo(x.key);
    if      (cmp  < 0)
       x.left  = put(x.left, key, val);
    else if (cmp  > 0)
       x.right = put(x.right, key, val);
    else
       if (cmp == 0)
           x.val = val;
    return x;
}
```

Cost. Number of compares is equal to 1 + depth of node.
Tree shape

- Many BSTs correspond to same set of keys.
- Number of compares for search/insert is equal to 1 + depth of node.

Remark. Tree shape depends on order of insertion.
**BST insertion: random order visualization**

**Ex.** Insert keys in random order.
Remark. Correspondence is 1-1 if array has no duplicate keys.
BSTs: mathematical analysis

**Proposition.** If \( N \) distinct keys are inserted into a BST in random order, the expected number of compares for a search/insert is \( \sim 2 \ln N \).

**Pf.** 1-1 correspondence with quicksort partitioning.

**Proposition.** [Reed, 2003] If \( N \) distinct keys are inserted in random order, expected height of tree is \( \sim 4.311 \ln N \).

**How Tall is a Tree?**

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**ABSTRACT**

Let \( H_n \) be the height of a random binary search tree on \( n \) nodes. We show that there exists constants \( \alpha = 4.31107 \ldots \) and \( \beta = 1.95 \ldots \) such that \( E(H_n) = \alpha \log n - \beta \log \log n + O(1) \), We also show that \( \text{Var}(H_n) = O(1) \).

**But...** Worst-case height is \( N \).

(exponentially small chance when keys are inserted in random order)
### ST implementations: summary

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<td>insert</td>
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<td>insert</td>
</tr>
<tr>
<td>sequential search (unordered list)</td>
<td>N</td>
<td>N</td>
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<td>N</td>
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<tr>
<td></td>
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<tr>
<td>binary search (ordered array)</td>
<td>lg N</td>
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<td>BST</td>
<td>N</td>
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<td>1.39 lg N</td>
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</tbody>
</table>
BSTs

- ordered operations
- deletion
Minimum and maximum

**Minimum.** Smallest key in table.
**Maximum.** Largest key in table.

**Q.** How to find the min / max?
Floor and ceiling

**Floor.** Largest key ≤ to a given key.

**Ceiling.** Smallest key ≥ to a given key.

Q. How to find the floor /ceiling?
Computing the floor

**Case 1.** \([k \text{ equals the key at root}]\)
The floor of \(k\) is \(k\).

**Case 2.** \([k \text{ is less than the key at root}]\)
The floor of \(k\) is in the left subtree.

**Case 3.** \([k \text{ is greater than the key at root}]\)
The floor of \(k\) is in the right subtree
(if there is any key \(\leq k\) in right subtree);
otherwise it is the key in the root.
Computing the floor

```java
public Key floor(Key key)
{
    Node x = floor(root, key);
    if (x == null) return null;
    return x.key;
}

private Node floor(Node x, Key key)
{
    if (x == null) return null;
    int cmp = key.compareTo(x.key);
    if (cmp == 0) return x;
    if (cmp < 0)  return floor(x.left, key);
    Node t = floor(x.right, key);
    if (t != null) return t;
    else           return x;
}
```

**finding floor(G)**
- G is less than S so floor(G) must be on the left
- G is greater than E so floor(G) could be on the right
- floor(G) in left subtree is null

**result**
Subtree counts

In each node, we store the number of nodes in the subtree rooted at that node. To implement size(), return the count at the root.

Remark. This facilitates efficient implementation of rank() and select().
BST implementation: subtree counts

private class Node
{
    private Key key;
    private Value val;
    private Node left;
    private Node right;
    private int N;
}

public int size()
{  return size(root);  }
private int size(Node x)
{  if (x == null) return 0;
   return x.N;
}

private Node put(Node x, Key key, Value val)
{  if (x == null) return new Node(key, val);
    int cmp = key.compareTo(x.key);
    if      (cmp  < 0) x.left  = put(x.left,  key, val);
    else if (cmp  > 0) x.right = put(x.right, key, val);
    else
        if (cmp == 0) x.val = val;
    x.N = 1 + size(x.left) + size(x.right);
    return x;
}
Rank

Rank. How many keys < \( k \)?

Easy recursive algorithm (4 cases!)

```
public int rank(Key key) {
    return rank(key, root);
}

private int rank(Key key, Node x) {
    if (x == null) return 0;
    int cmp = key.compareTo(x.key);
    if      (cmp  < 0) return rank(key, x.left);
    else if (cmp  > 0) return 1 + size(x.left) + rank(key, x.right);
    else if (cmp == 0) return size(x.left);
}
```
**Selection**

**Select. Key of given rank.**

```java
public Key select(int k) {
    if (k < 0) return null;
    if (k >= size()) return null;
    Node x = select(root, k);
    return x.key;
}

private Node select(Node x, int k) {
    if (x == null) return null;
    int t = size(x.left);
    if      (t  > k)
        return select(x.left,  k);
    else if (t  < k)
        return select(x.right, k-t-1);
    else if (t == k)
        return x;
    }
```
Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```java
public Iterable<Key> keys()
{
    Queue<Key> q = new Queue<Key>();
    inorder(root, q);
    return q;
}

private void inorder(Node x, Queue<Key> q)
{
    if (x == null) return;
    inorder(x.left, q);
    q.enqueue(x.key);
    inorder(x.right, q);
}
```

Property. Inorder traversal of a BST yields keys in ascending order.
Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```
inorder(S)
inorder(E)
inorder(A)
  enqueue A
  inorder(C)
    enqueue C
    enqueue E
    inorder(R)
      enqueue H
      inorder(M)
        enqueue M
        enqueue R
      enqueue S
  inorder(X)
    enqueue X
```
BST: ordered symbol table operations summary

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<th>Operation</th>
<th>Sequential Search</th>
<th>Binary Search</th>
<th>BST</th>
</tr>
</thead>
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<tr>
<td>search</td>
<td>$N$</td>
<td>$\lg N$</td>
<td>$h$</td>
</tr>
<tr>
<td>insert</td>
<td>$1$</td>
<td>$N$</td>
<td>$h$</td>
</tr>
<tr>
<td>min / max</td>
<td>$N$</td>
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<td>$h$</td>
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<td>floor / ceiling</td>
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</tr>
<tr>
<td>select</td>
<td>$N$</td>
<td>$1$</td>
<td>$h$</td>
</tr>
<tr>
<td>ordered iteration</td>
<td>$N \log N$</td>
<td>$N$</td>
<td>$N$</td>
</tr>
</tbody>
</table>

$h = \text{height of BST (proportional to } \log N \text{ if keys inserted in random order)}$

Order of growth of running time of ordered symbol table operations
BSTs
ordered operations
deletion
### ST implementations: summary

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**Next.** Deletion in BSTs.
BST deletion: lazy approach

To remove a node with a given key:
• Set its value to null.
• Leave key in tree to guide searches (but don't consider it equal to search key).

Cost. \( \sim 2 \ln N' \) per insert, search, and delete (if keys in random order), where \( N' \) is the number of key-value pairs ever inserted in the BST.

Unsatisfactory solution. Tombstone overload.
Deleting the minimum

To delete the minimum key:
• Go left until finding a node with a null left link.
• Replace that node by its right link.
• Update subtree counts.

```java
public void deleteMin()
{  root = deleteMin(root);  }

private Node deleteMin(Node x)
{
    if (x.left == null) return x.right;
    x.left = deleteMin(x.left);
    x.N = 1 + size(x.left) + size(x.right);
    return x;
}
```
Hibbard deletion

To delete a node with key $k$: search for node $t$ containing key $k$.

**Case 0.** [0 children] Delete $t$ by setting parent link to null.
Hibbard deletion

To delete a node with key $k$: search for node $t$ containing key $k$.

**Case 1.** [1 child] Delete $t$ by replacing parent link.
Hibbard deletion

To delete a node with key $k$: search for node $t$ containing key $k$.

**Case 2.** [2 children]
- Find successor $x$ of $t$.
- Delete the minimum in $t$'s right subtree.
- Put $x$ in $t$'s spot.

$x$ has no left child
but don't garbage collect $x$
still a BST
public void delete(Key key)
{  root = delete(root, key);  }

private Node delete(Node x, Key key) {
    if (x == null) return null;
    int cmp = key.compareTo(x.key);
    if (cmp < 0) x.left = delete(x.left, key);
    else if (cmp > 0) x.right = delete(x.right, key);
    else {
        if (x.right == null) return x.left;

        Node t = x;
        x = min(t.right);
        x.right = deleteMin(t.right);
        x.left = t.left;
    }
    x.N = size(x.left) + size(x.right) + 1;
    return x;
}
Hibbard deletion: analysis

Unsatisfactory solution. Not symmetric.

Surprising consequence. Trees not random (!) $\Rightarrow \sqrt{N}$ per op.

Longstanding open problem. Simple and efficient delete for BSTs.
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 Other operations also become √N if deletions allowed.

**Red-black BST.** *Guarantee* logarithmic performance for all operations.