- Define lambda
- Define ints
- is 1+1 = 2?
- contextual equivalence
- applicative bisimulation

1 Review of the Lambda Calculus

1.1 LAMBDA SYNTAX

x, y, z	Variables
$t, s, r ::=$	Terms
$\quad \lambda x.t$	Abstraction
$\quad x$	Variable
$\quad t\ s$	Application
$v, u ::=$	Values
$\quad \lambda x.t$	Abstraction
$\mathcal{E} ::=$	CBV Evaluation Contexts
$\quad \bullet$	Hole
$\quad \mathcal{E}\ s$	Application Left
$\quad v\ \mathcal{E}$	Application Right

Write $\mathcal{E}[t]$ for $\mathcal{E}[{}^{t}\!/\!_{\bullet}]$.

1.2 FREE VARIABLES $(fv(t) = \bar{x})$ where $\bar{x} = \{x_1, \ldots x_n\}$ and $\bar{y} = \{y_1, \ldots y_m\}$

$$\frac{}{fv(z) = \{z\}} \qquad \frac{fv(t) = \bar{x}}{fv(\lambda z.t) = \bar{x} \setminus \{z\}} \qquad \frac{fv(t) = \bar{x} \quad fv(s) = \bar{y}}{fv(t\ s) = \bar{x} \cup \bar{y}}$$

1.3 ALPHA EQUIVALENCE $(t \equiv s)$

$$\frac{}{x \equiv x} \qquad \frac{}{\lambda x.t \equiv \lambda y.t[{}^{y}\!/\!_{x}]}\ y \notin fv(t) \qquad \frac{t \equiv t' \quad s \equiv s'}{t\ s \equiv t'\ s'}$$

1.4 SUBSTITUTION $(t[{}^{r}\!/\!_{z}] = s)$

$$\frac{}{z[{}^{r}\!/\!_{z}] = r} \qquad \frac{}{x[{}^{r}\!/\!_{z}] = x}\ x \neq z \qquad \frac{t[{}^{r}\!/\!_{z}] = t'}{(\lambda x.t)[{}^{r}\!/\!_{z}] = \lambda x.t'}\ \begin{array}{l} x \neq z \\ x \notin fv(r) \end{array} \qquad \frac{t[{}^{r}\!/\!_{z}] = t' \quad s[{}^{r}\!/\!_{z}] = s'}{(t\ s)[{}^{r}\!/\!_{z}] = t'\ s'}$$

1.5 EVALUATION $(t \rightarrow s)$

$$\frac{}{(\lambda x.t)\ s \rightarrow t[{}^{s}\!/\!_{x}]} \qquad \frac{t \rightarrow s}{\mathcal{E}[t] \rightarrow \mathcal{E}[s]} \qquad \frac{t \equiv t' \quad t' \rightarrow s' \quad s' \equiv s}{t \rightarrow s}$$

Alternative to using alpha equivalence in evaluation, is to identify syntax up to alpha equivalence, or to make substitution total.

$$\frac{(\lambda x\,.\,t) \equiv (\lambda y\,.\,s) \quad s[{}^{r}\!/\!_{z}] = s'}{(\lambda x\,.\,t)[{}^{r}\!/\!_{z}] = \lambda y\,.\,s'}\;\begin{matrix} y \neq z \\ y \notin fv(r) \end{matrix}$$

1.6 EVALUATION $(t \Rightarrow s)$

$$\frac{}{t \Rightarrow t} \qquad \frac{t \rightarrow s \quad s \Rightarrow r}{t \Rightarrow r}$$

2 Other Semantics

2.1 Big Step Semantics

2.1 BIG-STEP EVALUATION $(t \Downarrow v)$

$$\frac{}{v \Downarrow v} \qquad \frac{t \Downarrow \lambda x\,.\,r \quad r[{}^{s}\!/\!_{x}] \Downarrow v}{t\,s \Downarrow v} \qquad \frac{t \equiv t' \quad t' \Downarrow v}{t \Downarrow v}$$

PROPOSITION 2.2. $t \Downarrow v$ *implies* $t \Rightarrow v$.

PROPOSITION 2.3. $t \Rightarrow v$ *implies* $t \Downarrow v$.

2.2 Explicit Stack

2.4 STACKS

σ, ρ ::=	Stacks
$\quad \varepsilon$	Empty
$\quad \sigma, x{=}t$	Element
$\mathcal{E}$ ::=	CBV Evaluation Contexts
$\quad \bullet$	Hole
$\quad \mathcal{E}\,s$	Application Left

2.5 STACK LOOKUP $(\sigma(x) = t)$

$$\frac{}{(\sigma, x{=}t)(x) = t} \qquad \frac{\sigma(x) = t}{(\sigma, y{=}s)(x) = t}\; x \neq y$$

2.6 STACK EVALUATION $(\sigma \triangleright t \rightarrow s)$

$$\frac{\sigma, x{=}s \triangleright t \rightarrow r}{\sigma \triangleright (\lambda x\,.\,t)\,s \rightarrow r} \qquad \frac{\sigma \triangleright t \rightarrow s}{\sigma \triangleright \mathcal{E}[t] \rightarrow \mathcal{E}[s]} \qquad \frac{\sigma(x) = s}{\sigma \triangleright x \rightarrow s}$$

PROPOSITION 2.7. $\varepsilon \triangleright t \Rightarrow v$ *implies* $t \Rightarrow v$.

PROPOSITION 2.8. $t \Rightarrow v$ *implies* $\varepsilon \triangleright t \Rightarrow v$.

2.3 SECD Machine

2.4 Computational Lambda Calculus

3 Contextual Equivalence

3.1 CONVERGENCE $(\mathsf{t}\Downarrow)$

$$\frac{}{\mathsf{v}\Downarrow} \qquad \frac{\mathsf{t} \to \mathsf{s} \quad \mathsf{s}\Downarrow}{\mathsf{t}\Downarrow}$$

3.2 CONTEXTUAL EQUIVALENCE $(\mathsf{t} \approx \mathsf{s})$

Terms t and s are *equivalent* (notation $\mathsf{t} \approx \mathsf{s}$) if for all contexts $\mathcal{E}$: $\mathcal{E}[\mathsf{t}]\Downarrow$ iff $\mathcal{E}[\mathsf{s}]\Downarrow$.

3.3 CHURCH NUMERALS

$\mathtt{c}_0 \triangleq \lambda\mathtt{s}\,.\,\lambda\mathtt{z}\,.\,\mathtt{z}$
$\mathtt{c}_1 \triangleq \lambda\mathtt{s}\,.\,\lambda\mathtt{z}\,.\,\mathtt{s}\;\mathtt{z}$
$\mathtt{c}_2 \triangleq \lambda\mathtt{s}\,.\,\lambda\mathtt{z}\,.\,\mathtt{s}\;(\mathtt{s}\;\mathtt{z})$
$\quad\ldots$
$\mathtt{succ} \triangleq \lambda\mathtt{n}\,.\,\lambda\mathtt{s}\,.\,\lambda\mathtt{z}\,.\,\mathtt{s}\;(\mathtt{n}\;\mathtt{s}\;\mathtt{z})$

Prove that $(\mathtt{succ}\;\mathtt{c}_1) \approx \mathtt{c}_2$

$$\begin{aligned} \mathtt{succ}\;\mathtt{c}_1 &\triangleq (\lambda\mathtt{n}\,.\,\lambda\mathtt{s}\,.\,\lambda\mathtt{z}\,.\,\mathtt{s}\;(\mathtt{n}\;\mathtt{s}\;\mathtt{z}))\;(\lambda\mathtt{s}\,.\,\lambda\mathtt{z}\,.\,\mathtt{s}\;\mathtt{z}) \\ &\to \lambda\mathtt{s}\,.\,\lambda\mathtt{z}\,.\,\mathtt{s}\;((\lambda\mathtt{s}'\,.\,\lambda\mathtt{z}'\,.\,\mathtt{s}'\;\mathtt{z}')\;\mathtt{s}\;\mathtt{z}) \end{aligned}$$

Let U be some universal set and $F : 2^U \to 2^U$ be a monotone function. Induction and co-induction are dual proof principles that derive from the definition of a set to be the least or greatest solution, respectively, of equations of the form $X = F(X)$. First some definitions.

3.4 FIXPOINTS, F-CLOSED AND F-DENSE SETS

A function $F : 2^U \to 2^U$ is *monotone* if $X \subseteq Y$ implies $F(X) \subseteq F(Y)$.
A *fixpoint* of F is a solution of the equation $X = F(X)$.
A set $X \subseteq U$ is *F-closed* iff $F(X) \subseteq X$.
A set $X \subseteq U$ is *F-dense* iff $X \subseteq F(X)$.
$\mu X.F(X) \triangleq \bigcap\{X \mid F(X) \subseteq X\}$.
$\nu X.F(X) \triangleq \bigcup\{X \mid X \subseteq F(X)\}$.

LEMMA 3.5.

(1) $\mu X.F(X)$ is the least F-closed set.
(2) $\nu X.F(X)$ is the greatest F-dense set.

Proof. We prove (2); (1) follows by a dual argument. Since $\nu X.F(X)$ contains every F-dense set by construction, we need only show that it is itself F-dense, for which the following lemma suffices.

If every X_i is F-dense, so is the union $\bigcup_i X_i$.

Since $X_i \subseteq F(X_i)$ for every i, $\bigcup_i X_i \subseteq \bigcup_i F(X_i)$. Since F is monotone, $F(X_i) \subseteq F(\bigcup_i X_i)$ for each i. Therefore $\bigcup_i F(X_i) \subseteq F(\bigcup_i X_i)$, and so we have $\bigcup_i X_i \subseteq F(\bigcup_i X_i)$ by transitivity, that is, $\bigcup_i X_i$ is F-dense. $\square$

THEOREM 3.6 (TARSKI).

(1) $\mu X.F(X)$ *is the least fixpoint of* F.

(2) $\nu X.F(X)$ *is the greatest fixpoint of* F.

Proof. Again we prove (2) alone; (1) follows by a dual argument. Let $\overline{\nu} = \nu X.F(X)$. We have $\overline{\nu} \subseteq F(\overline{\nu})$ by Lemma 3.5. So $F(\overline{\nu}) \subseteq F(F(\overline{\nu}))$ by monotonicity of F. But then $F(\overline{\nu})$ is F-dense, and therefore $F(\overline{\nu}) \subseteq \overline{\nu}$. Combining the inequalities we have $\overline{\nu} = F(\overline{\nu})$; it is the greatest fixpoint because any other is F-dense, and hence contained in $\overline{\nu}$. $\square$

We obtain two dual methods for defining sets and dual proof principles associated with these definitions.

3.7 INDUCTION AND COINDUCTION

$\mu X.F(X)$, the least solution of $X = F(X)$, is the set *inductively defined* by F.
$\nu X.F(X)$, the greatest solution of $X = F(X)$, is the set *co-inductively defined* by F.
Principle of induction: If X is F-closed then $\mu X.F(X) \subseteq X$.
Principle of co-induction: If X is F-dense then $X \subseteq \nu X.F(X)$.

Mathematical induction is a special case. Suppose there is an element $0 \in U$ and an injective function $S : U \to U$. If we define a monotone function $F : 2^U \to 2^U$ by

$$F(X) \triangleq \{0\} \cup \{S(x) \mid x \in X\}$$

and set $\mathbb{N} \triangleq \mu X.F(X)$, the associated principle of induction is that $\mathbb{N} \subseteq X$ if $F(X) \subseteq X$, which is to say that $\mathbb{N} \subseteq X$ if both $0 \in X$ and $\forall x \in X.(S(x) \in X)$.

PROPOSITION 3.8. *Let U be an arbitrary universal set and let $F : 2^U \to 2^U$ be some monotone function. If $\overline{\nu} \triangleq \nu X.F(X)$ we have:*

$$\begin{aligned} \overline{\nu} &= \nu X.F(X) \cup \overline{\nu} && (\nu.\mathrm{I}) \\ &= \nu X.F(X \cup \overline{\nu}) && (\nu.\mathrm{II}) \\ &= \nu X.F(X \cup \overline{\nu}) \cup \overline{\nu} && (\nu.\mathrm{III}) \end{aligned}$$

Proof. Make the following definitions.

$$\begin{aligned} \overline{\nu}_1 &\triangleq \nu X.F(X) \cup \overline{\nu} \\ \overline{\nu}_2 &\triangleq \nu X.F(X \cup \overline{\nu}) \\ \overline{\nu}_3 &\triangleq \nu X.F(X \cup \overline{\nu}) \cup \overline{\nu} \end{aligned}$$

We must show that each $\overline{\nu}_i$ equals $\overline{\nu}$. Since $\overline{\nu} \subseteq F(\overline{\nu})$ it follows by coinduction that $\overline{\nu} \subseteq \overline{\nu}_i$ for each i. The reverse inclusions take a little more work.

$\overline{\nu}_2 \subseteq \overline{\nu}$. Since $\overline{\nu}$ is a fixpoint of F, which is monotone, we have $\overline{\nu} = F(\overline{\nu}) \subseteq F(\overline{\nu}_2 \cup \overline{\nu})$. Now $\overline{\nu}_2 \subseteq F(\overline{\nu}_2 \cup \overline{\nu})$ so $\overline{\nu}_2 \cup \overline{\nu} \subseteq F(\overline{\nu}_2 \cup \overline{\nu})$, and therefore $\overline{\nu}_2 \cup \overline{\nu} \subseteq \overline{\nu}$ by co-induction. Hence $\overline{\nu}_2 \subseteq \overline{\nu}$.

$\overline{\nu}_1 \subseteq \overline{\nu}_2$. We have $\overline{\nu}_1 = F(\overline{\nu}_1) \cup \overline{\nu} = F(\overline{\nu}_1) \cup F(\overline{\nu}) \subseteq F(\overline{\nu}_1 \cup \overline{\nu})$. So $\overline{\nu}_1 \subseteq \nu X.F(X \cup \overline{\nu})$.

$\overline{\nu}_3 \subseteq \overline{\nu}_2$. We have $\overline{\nu}_3 = F(\overline{\nu}_3 \cup \overline{\nu}) \cup \overline{\nu} = F(\overline{\nu}_3 \cup \overline{\nu}) \cup F(\overline{\nu}) = F(\overline{\nu}_3 \cup \overline{\nu})$ since $F(\overline{\nu}) \subseteq F(\overline{\nu}_3 \cup \overline{\nu})$. Hence $\overline{\nu}_3 \subseteq \nu X.F(X \cup \overline{\nu})$. □

3.9 STRONG VERSIONS (WHERE $\overline{\mu} = \mu X.F(X)$ AND $\overline{\nu} = \nu X.F(X)$)

Strong induction: If $F(X \cap \overline{\mu}) \cap \overline{\mu} \subseteq X$ then $\overline{\mu} \subseteq X$.
Strong co-induction: If $X \subseteq F(X \cup \overline{\nu}) \cup \overline{\nu}$ then $\overline{\nu} \subseteq X$.

For numbers, our strong induction yields $\mathbb{N} \subseteq X$ if $\{0\} \cup \{S(x) \in \mathbb{N} \mid x \in X \wedge x \in \mathbb{N}\} \subseteq X$.

If $0 \in X$ and $S(x) \in X$ for every $x \in \mathbb{N} \cap X$, then $\mathbb{N} \subseteq X$.

4 Applicative Bisimulation