- Define lambda
- Define ints
- is 1+1 = 2?
- contextual equivalence
- applicative bisimulation

# 1 Review of the Lambda Calculus

1.1 LAMBDA SYNTAX

| | | |
|---|---|---|
| $x, y, z$ | Variables | |
| $t, s, r ::=$ | Terms | |
| $\lambda x . t$ | | Abstraction |
| $x$ | | Variable |
| $t\ s$ | | Application |
| $v, u ::=$ | Values | |
| $\lambda x . t$ | | Abstraction |
| $\mathcal{E} ::=$ | CBV Evaluation Contexts | |
| $\bullet$ | | Hole |
| $\mathcal{E}\ s$ | | Application Left |
| $v\ \mathcal{E}$ | | Application Right |

Write $\mathcal{E}[t]$ for $\mathcal{E}[{}^{t}/_{\bullet}]$.

1.2 FREE VARIABLES $(fv(t) = \bar{x})$ where $\bar{x} = \{x_1, \dots x_n\}$ and $\bar{y} = \{y_1, \dots y_m\}$

$$\frac{}{fv(z) = \{z\}} \qquad \frac{fv(t) = \bar{x}}{fv(\lambda z . t) = \bar{x} \setminus \{z\}} \qquad \frac{fv(t) = \bar{x} \quad fv(s) = \bar{y}}{fv(t\ s) = \bar{x} \cup \bar{y}}$$

1.3 ALPHA EQUIVALENCE $(t \equiv s)$

$$\frac{}{x \equiv x} \qquad \frac{}{\lambda x . t \equiv \lambda y . t[{}^{y}/_{x}]}\ y \notin fv(t) \qquad \frac{t \equiv t' \quad s \equiv s'}{t\ s \equiv t'\ s'}$$

1.4 SUBSTITUTION $(t[{}^{r}/_{z}] = s)$

$$\frac{}{z[{}^{r}/_{z}] = r} \qquad \frac{}{x[{}^{r}/_{z}] = x}\ x \neq z \qquad \frac{t[{}^{r}/_{z}] = t'}{(\lambda x . t)[{}^{r}/_{z}] = \lambda x . t'}\ \begin{matrix} x \neq z \\ x \notin fv(r) \end{matrix} \qquad \frac{t[{}^{r}/_{z}] = t' \quad s[{}^{r}/_{z}] = s'}{(t\ s)[{}^{r}/_{z}] = t'\ s'}$$

1.5 EVALUATION $(t \to s)$

$$\frac{}{(\lambda x . t)\ s \to t[{}^{s}/_{x}]} \qquad \frac{t \to s}{\mathcal{E}[t] \to \mathcal{E}[s]} \qquad \frac{t \equiv t' \quad t' \to s' \quad s' \equiv s}{t \to s}$$

Alternative to using alpha equivalence in evaluation, is to identify syntax up to alpha



equivalence, or to make substitution total.

$$\frac{(\lambda x . t) \equiv (\lambda y . s) \quad s[{}^{r}/_{z}] = s'}{(\lambda x . t)[{}^{r}/_{z}] = \lambda y . s'}\ \begin{matrix} y \neq z \\ y \notin fv(r) \end{matrix}$$

1.6 EVALUATION $(t \Rightarrow s)$

$$\frac{}{t \Rightarrow t} \qquad \frac{t \to s \quad s \Rightarrow r}{t \Rightarrow r}$$

# 2 Other Semantics

## 2.1 Big Step Semantics

2.1 BIG-STEP EVALUATION $(t \Downarrow v)$

$$\frac{}{v \Downarrow v} \qquad \frac{t \Downarrow \lambda x . r \quad r[{}^{s}/_{x}] \Downarrow v}{t\ s \Downarrow v} \qquad \frac{t \equiv t' \quad t' \Downarrow v}{t \Downarrow v}$$

PROPOSITION 2.2. $t \Downarrow v$ *implies* $t \Rightarrow v$.

PROPOSITION 2.3. $t \Rightarrow v$ *implies* $t \Downarrow v$.

## 2.2 Explicit Stack

2.4 STACKS

| | | |
|---|---|---|
| $\sigma, \rho ::=$ | Stacks | |
| $\varepsilon$ | | Empty |
| $\sigma, x{=}t$ | | Element |
| $\mathcal{E} ::=$ | CBV Evaluation Contexts | |
| $\bullet$ | | Hole |
| $\mathcal{E}\ s$ | | Application Left |

2.5 STACK LOOKUP $(\sigma(x) = t)$

$$\frac{}{(\sigma, x{=}t)(x) = t} \qquad \frac{\sigma(x) = t}{(\sigma, y{=}s)(x) = t}\ x \neq y$$

2.6 STACK EVALUATION $(\sigma \triangleright t \to s)$

$$\frac{\sigma, x{=}s \triangleright t \to r}{\sigma \triangleright (\lambda x . t)\ s \to r} \qquad \frac{\sigma \triangleright t \to s}{\sigma \triangleright \mathcal{E}[t] \to \mathcal{E}[s]} \qquad \frac{\sigma(x) = s}{\sigma \triangleright x \to s}$$

PROPOSITION 2.7. $\varepsilon \triangleright t \Rightarrow v$ *implies* $t \Rightarrow v$.

PROPOSITION 2.8. $t \Rightarrow v$ *implies* $\varepsilon \triangleright t \Rightarrow v$.



## 2.3 SECD Machine

## 2.4 Computational Lambda Calculus

# 3 Contextual Equivalence

3.1 CONVERGENCE $(t \Downarrow)$

$$\frac{}{v \Downarrow} \qquad \frac{t \to s \quad s \Downarrow}{t \Downarrow}$$

3.2 CONTEXTUAL EQUIVALENCE $(t \approx s)$

Terms $t$ and $s$ are *equivalent* (notation $t \approx s$) if for all contexts $\mathcal{E}$: $\mathcal{E}[t] \Downarrow$ iff $\mathcal{E}[s] \Downarrow$.

3.3 CHURCH NUMERALS

$c_0 \triangleq \lambda s . \lambda z . z$
$c_1 \triangleq \lambda s . \lambda z . s\ z$
$c_2 \triangleq \lambda s . \lambda z . s\ (s\ z)$
...
$\mathtt{succ} \triangleq \lambda n . \lambda s . \lambda z . s\ (n\ s\ z)$

Prove that $(\mathtt{succ}\ c_1) \approx c_2$

$$\begin{aligned} \mathtt{succ}\ c_1 &\triangleq (\lambda n . \lambda s . \lambda z . s\ (n\ s\ z))\ (\lambda s . \lambda z . s\ z) \\ &\to \lambda s . \lambda z . s\ ((\lambda s' . \lambda z' . s'\ z')\ s\ z) \end{aligned}$$

Let $U$ be some universal set and $F : 2^U \to 2^U$ be a monotone function. Induction and co-induction are dual proof principles that derive from the definition of a set to be the least or greatest solution, respectively, of equations of the form $X = F(X)$. First some definitions.

3.4 FIXPOINTS, $F$-CLOSED AND $F$-DENSE SETS

A function $F : 2^U \to 2^U$ is *monotone* if $X \subseteq Y$ implies $F(X) \subseteq F(Y)$.
A *fixpoint* of $F$ is a solution of the equation $X = F(X)$.
A set $X \subseteq U$ is *$F$-closed* iff $F(X) \subseteq X$.
A set $X \subseteq U$ is *$F$-dense* iff $X \subseteq F(X)$.
$\mu X.F(X) \triangleq \bigcap\{X \mid F(X) \subseteq X\}$.
$\nu X.F(X) \triangleq \bigcup\{X \mid X \subseteq F(X)\}$.

LEMMA 3.5.

*(1) $\mu X.F(X)$ is the least $F$-closed set.*
*(2) $\nu X.F(X)$ is the greatest $F$-dense set.*



*Proof.* We prove (2); (1) follows by a dual argument. Since $\nu X.F(X)$ contains every $F$-dense set by construction, we need only show that it is itself $F$-dense, for which the following lemma suffices.

If every $X_i$ is $F$-dense, so is the union $\bigcup_i X_i$.

Since $X_i \subseteq F(X_i)$ for every $i$, $\bigcup_i X_i \subseteq \bigcup_i F(X_i)$. Since $F$ is monotone, $F(X_i) \subseteq F(\bigcup_i X_i)$ for each $i$. Therefore $\bigcup_i F(X_i) \subseteq F(\bigcup_i X_i)$, and so we have $\bigcup_i X_i \subseteq F(\bigcup_i X_i)$ by transitivity, that is, $\bigcup_i X_i$ is F-dense. □

THEOREM 3.6 (TARSKI).

*(1) $\mu X.F(X)$ is the least fixpoint of $F$.*
*(2) $\nu X.F(X)$ is the greatest fixpoint of $F$.*

*Proof.* Again we prove (2) alone; (1) follows by a dual argument. Let $\bar{\nu} = \nu X.F(X)$. We have $\bar{\nu} \subseteq F(\bar{\nu})$ by Lemma 3.5. So $F(\bar{\nu}) \subseteq F(F(\bar{\nu}))$ by monotonicity of $F$. But then $F(\bar{\nu})$ is $F$-dense, and therefore $F(\bar{\nu}) \subseteq \bar{\nu}$. Combining the inequalities we have $\bar{\nu} = F(\bar{\nu})$; it is the greatest fixpoint because any other is $F$-dense, and hence contained in $\bar{\nu}$. □

We obtain two dual methods for defining sets and dual proof principles associated with these definitions.

3.7 INDUCTION AND COINDUCTION

$\mu X.F(X)$, the least solution of $X = F(X)$, is the set *inductively defined* by $F$.
$\nu X.F(X)$, the greatest solution of $X = F(X)$, is the set *co-inductively defined* by $F$.
*Principle of induction:* If $X$ is $F$-closed then $\mu X.F(X) \subseteq X$.
*Principle of co-induction:* If $X$ is $F$-dense then $X \subseteq \nu X.F(X)$.

Mathematical induction is a special case. Suppose there is an element $0 \in U$ and an injective function $S : U \to U$. If we define a monotone function $F : 2^U \to 2^U$ by

$$F(X) \triangleq \{0\} \cup \{S(x) \mid x \in X\}$$

and set $\mathbb{N} \triangleq \mu X.F(X)$, the associated principle of induction is that $\mathbb{N} \subseteq X$ if $F(X) \subseteq X$, which is to say that $\mathbb{N} \subseteq X$ if both $0 \in X$ and $\forall x \in X.(S(x) \in X)$.

PROPOSITION 3.8. *Let $U$ be an arbitrary universal set and let $F : 2^U \to 2^U$ be some monotone function. If $\bar{\nu} \triangleq \nu X.F(X)$ we have:*

$$\begin{aligned} \bar{\nu} &= \nu X.F(X) \cup \bar{\nu} && (\nu.\mathrm{I}) \\ &= \nu X.F(X \cup \bar{\nu}) && (\nu.\mathrm{II}) \\ &= \nu X.F(X \cup \bar{\nu}) \cup \bar{\nu} && (\nu.\mathrm{III}) \end{aligned}$$

*Proof.* Make the following definitions.

$\bar{\nu}_1 \triangleq \nu X.F(X) \cup \bar{\nu}$
$\bar{\nu}_2 \triangleq \nu X.F(X \cup \bar{\nu})$
$\bar{\nu}_3 \triangleq \nu X.F(X \cup \bar{\nu}) \cup \bar{\nu}$

We must show that each $\overline{\nu}_i$ equals $\overline{\nu}$. Since $\overline{\nu} \subseteq F(\overline{\nu})$ it follows by coinduction that $\overline{\nu} \subseteq \overline{\nu}_i$ for each $i$. The reverse inclusions take a little more work.

**$\overline{\nu}_2 \subseteq \overline{\nu}$.** Since $\overline{\nu}$ is a fixpoint of $F$, which is monotone, we have $\overline{\nu} = F(\overline{\nu}) \subseteq F(\overline{\nu}_2 \cup \overline{\nu})$. Now $\overline{\nu}_2 \subseteq F(\overline{\nu}_2 \cup \overline{\nu})$ so $\overline{\nu}_2 \cup \overline{\nu} \subseteq F(\overline{\nu}_2 \cup \overline{\nu})$, and therefore $\overline{\nu}_2 \cup \overline{\nu} \subseteq \overline{\nu}$ by co-induction. Hence $\overline{\nu}_2 \subseteq \overline{\nu}$.

**$\overline{\nu}_1 \subseteq \overline{\nu}_2$.** We have $\overline{\nu}_1 = F(\overline{\nu}_1) \cup \overline{\nu} = F(\overline{\nu}_1) \cup F(\overline{\nu}) \subseteq F(\overline{\nu}_1 \cup \overline{\nu})$. So $\overline{\nu}_1 \subseteq \nu X.F(X \cup \overline{\nu})$.

**$\overline{\nu}_3 \subseteq \overline{\nu}_2$.** We have $\overline{\nu}_3 = F(\overline{\nu}_3 \cup \overline{\nu}) \cup \overline{\nu} = F(\overline{\nu}_3 \cup \overline{\nu}) \cup F(\overline{\nu}) = F(\overline{\nu}_3 \cup \overline{\nu})$ since $F(\overline{\nu}) \subseteq F(\overline{\nu}_3 \cup \overline{\nu})$. Hence $\overline{\nu}_3 \subseteq \nu X.F(X \cup \overline{\nu})$. □

3.9 STRONG VERSIONS (WHERE $\overline{\mu} = \mu X.F(X)$ AND $\overline{\nu} = \nu X.F(X)$)

*Strong induction:* If $F(X \cap \overline{\mu}) \cap \overline{\mu} \subseteq X$ then $\overline{\mu} \subseteq X$.
*Strong co-induction:* If $X \subseteq F(X \cup \overline{\nu}) \cup \overline{\nu}$ then $\overline{\nu} \subseteq X$.

For numbers, our strong induction yields $\mathbb{N} \subseteq X$ if $\{0\} \cup \{S(x) \in \mathbb{N} \mid x \in X \wedge x \in \mathbb{N}\} \subseteq X$.

If $0 \in X$ and $S(x) \in X$ for every $x \in \mathbb{N} \cap X$, then $\mathbb{N} \subseteq X$.

# 4 Applicative Bisimulation